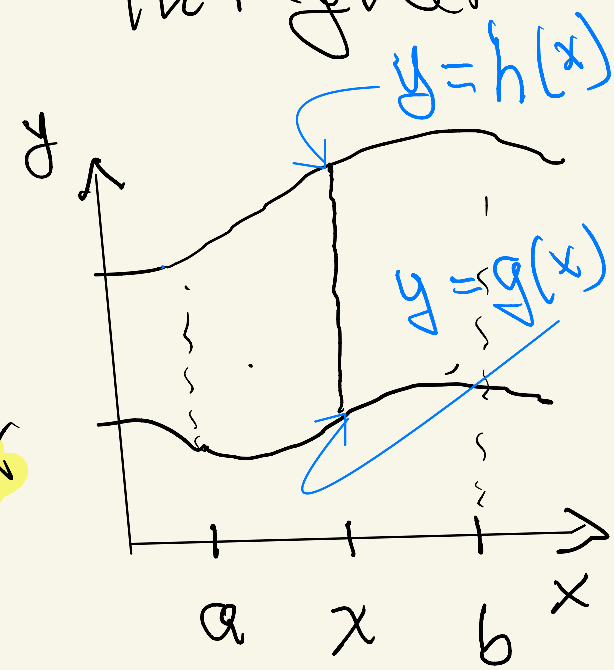
§15.3 Polar Coordinates

- Recall the defn of integral:

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j)\in R} f(x_i,y_j)\Delta x \Delta y}_{\text{2-D Riemann Sum}}$$

- To evaluate: iterate the integral

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

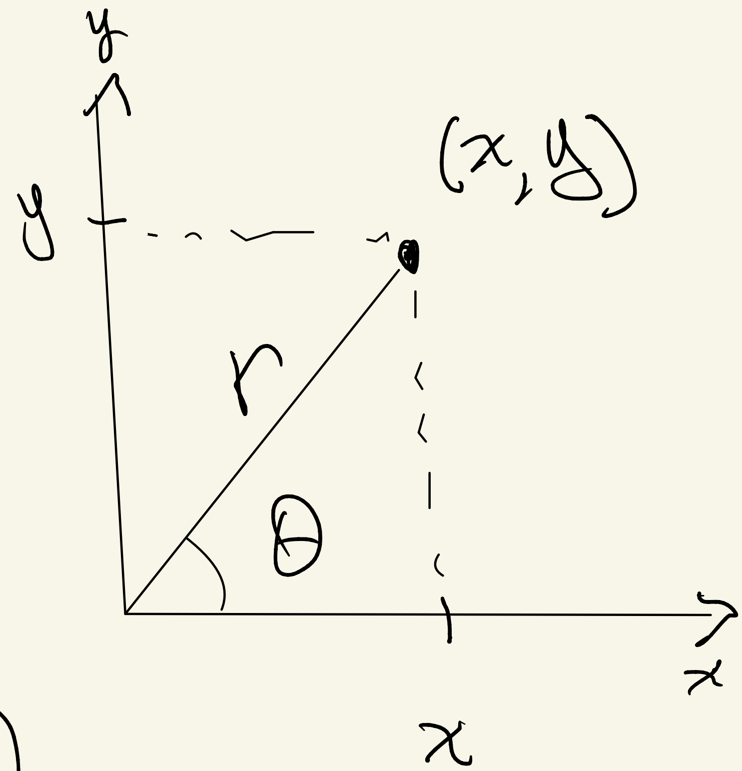


- when the function f is radially symmetric the integral can be evaluated more easily in polar coordinates
- The idea: change variables from $(x,y) \longrightarrow (r,\theta)$, and then write the Riemann Sum in terms of r and θ ...

Ie $$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{R_{r\theta}} \tilde{f}(r_i,\theta_j)\,\Delta r\,d\theta}_{\text{Riemann Sum in } (r,\theta)}$$

- Recall the expression for x and y in terms of r and θ:

$$x = r\cos\theta$$
$$y = r\sin\theta$$



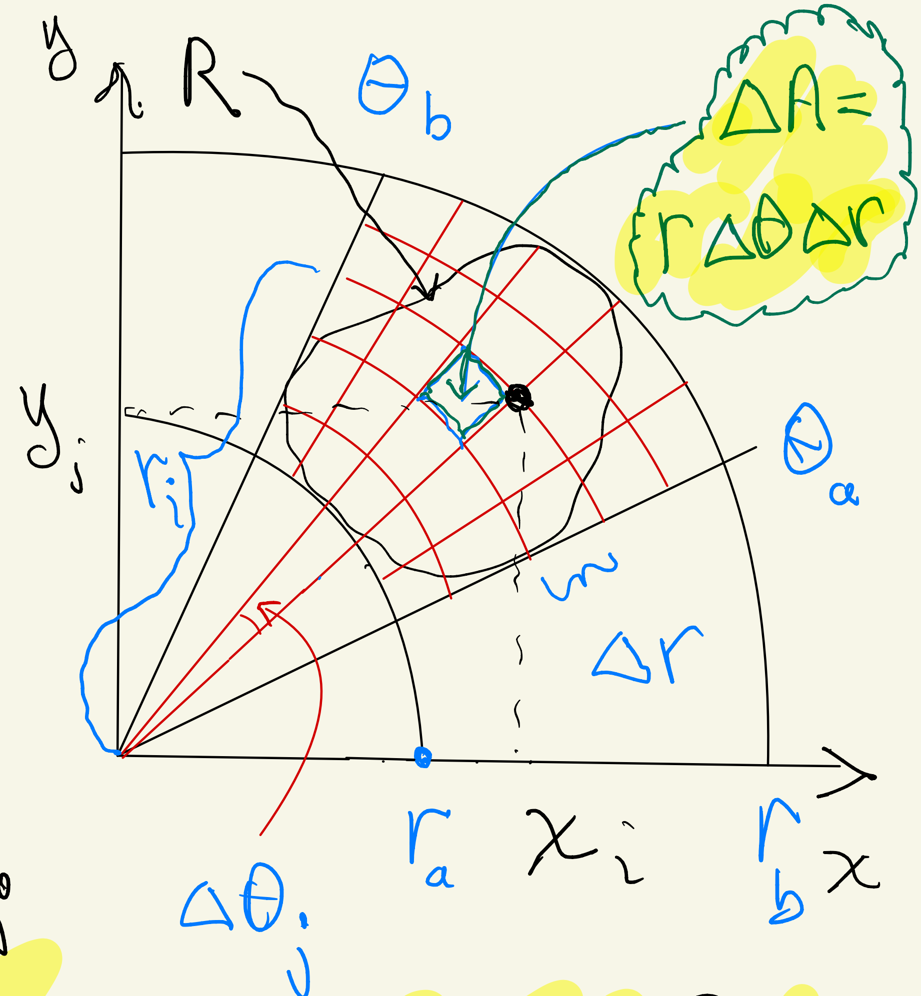
- Thus its easy to write $f(x,y)$ in terms of r & θ:

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \bar{f}(r,\theta)$$

- Q: How does the area change betw $\Delta x \Delta y$ & $\Delta r \Delta\theta$?

- So consider the problem of evaluating $\iint_R f(x,y)\,dA$ in polar coordinates —

To do this we write integral as a Riemann Sum in (r,θ)



$$x_i = r_i \cos\theta_j$$

$$y_j = r_i \cos\theta_j$$

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{(r_i,\theta_j)\in R_{r\theta}} \tilde{f}(r_i,\theta_j)\,\Delta r\,\Delta\theta}_{\text{Riemann Sum in } r,\theta}$$

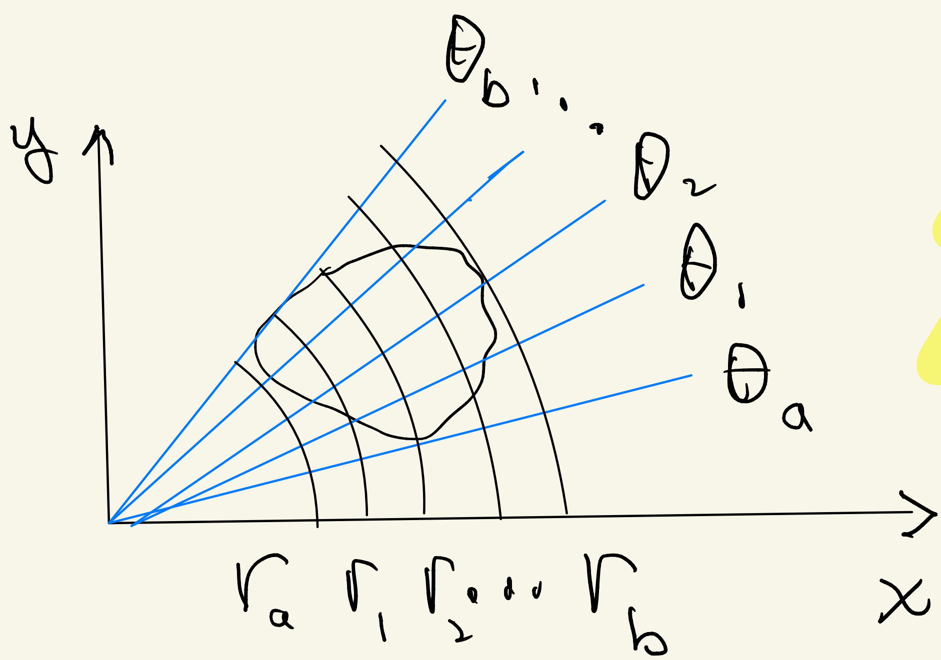
• That is: Draw the region $R_{r\theta}$ in xy-coordinates, cover it with a grid

$$r_a = r_0 < r_1 < \cdots < r_N = r_b$$

$$\theta_a = \theta_0 < \theta_1 < \cdots < \theta_N = \theta_b$$

$$\Delta r = \frac{r_b - r_a}{N}, \quad r_i = r_a + i\,\Delta r$$

$$\Delta\theta = \frac{\theta_b - \theta_a}{N}, \quad \theta_j = \theta_a + j\,\Delta\theta$$



View $R_{r\theta}$ in the xy-plane.

- **Main question**: what is the amplification factor?

I.e., how much must area $\Delta r \Delta \theta$ in (r,θ)-plane be multiplied to give its area in the (x,y)-plane?

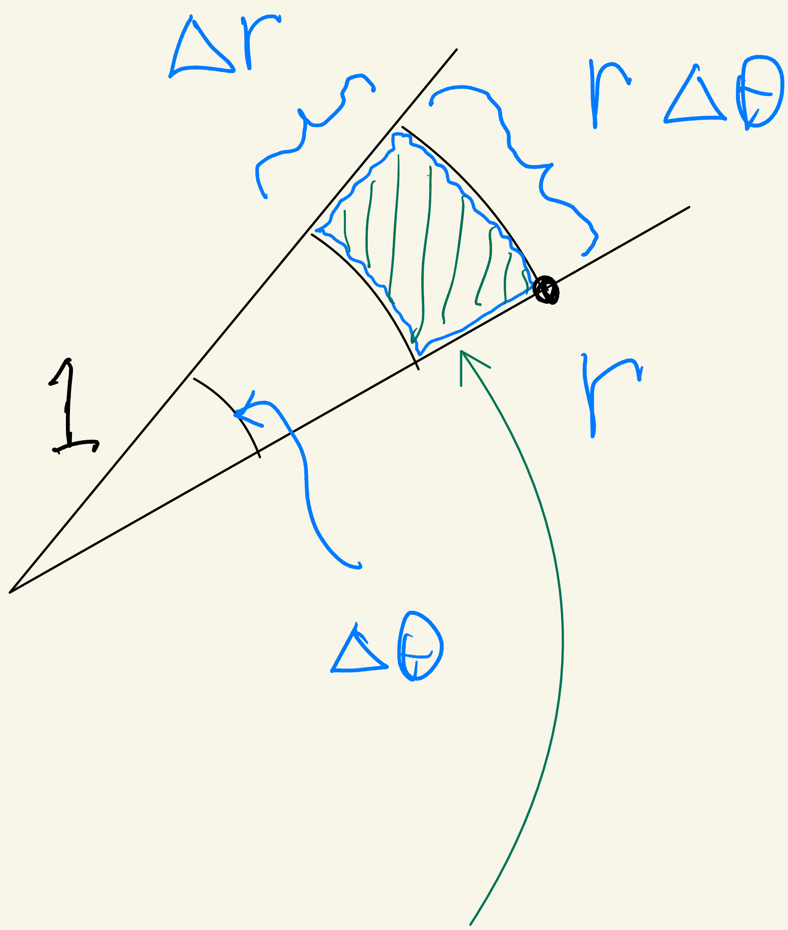
That is:

$$\Delta x \Delta y = \boxed{?} \, \Delta r \Delta \theta$$

Amplification factor for area

Ans: We get this from the geometry

- To get amplification factor blow up the picture —



Area in the (x,y)-plane is $\Delta A = r\,\Delta r\,\Delta\theta$

Amplification factor $= r$

- **Conclude:**

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \sum_{(x_i,y_j)\in R_{xy}} f(x_i,y_j)\,\Delta x\,\Delta y$$

$$= \lim_{N\to\infty} \underbrace{\sum_{(r_i,\theta_j)\in R_{r\theta}} \underbrace{f(r_i\cos\theta_j,\, r_i\sin\theta_j)}_{\tilde{f}(r_i,\theta_j)}\, \underbrace{r_i\,\Delta r\,\Delta\theta}_{\Delta A}}_{\text{Riemann Sum in } (r,\theta)}$$

$$= \iint_{R_{r\theta}} \overbrace{\underbrace{f(r\cos\theta,\, r\sin\theta)}_{\tilde{f}(r,\theta)}}^{\tilde{f}(r,\theta)}\, r\,dr\,d\theta$$

requires amplification factor

- **Key Take Away**: We are interested in evaluating an integral in (x,y)-coordinates

We express the function and draw the region in (x,y)-coords

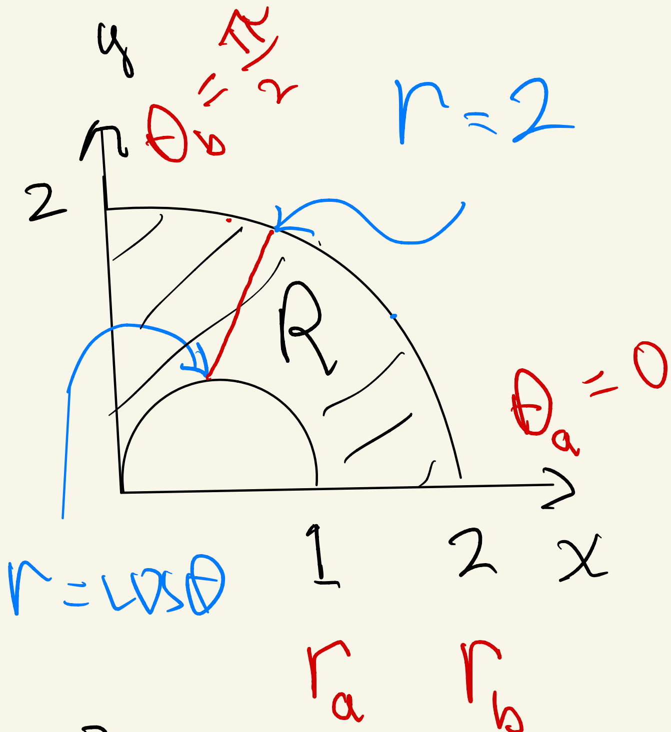
We write the grid in (r,θ) & express a volume element in (r,θ)

$$\Delta V_{ij} = f(r_i\cos\theta, r_i\sin\theta)\underbrace{r_i\Delta r\Delta\theta}_{\text{"}\Delta x\Delta y\text{"}}$$

$$\iint_R f(x,y)\,dA = \sum_R \Delta V_{ij} = \iint_{R_{r\theta}} \bar{f}(r,\theta)\,r\,dr\,d\theta$$

Ex **Example ①** Find the mass M of a metal plate of constant density $\delta(x,y) = \delta = \text{Const}$ that lies betw $r=2$, $r=\cos\theta$, $0 \le \theta \le \pi/2$

Soln: Picture



$$\text{Mass} = \iint_R \delta \, dA$$

$$= \delta \iint_{R_{xy}} dx\,dy$$

$$= \delta \iint_{R_{r\theta}} r\,dr\,d\theta = \delta \int_0^{\pi/2} \int_{\cos\theta}^{2} r\,dr\,d\theta$$

Mass $= \delta \displaystyle\int_0^{\pi/2}\int_{\cos\theta}^{2} r\,dr\,d\theta$

$$= \delta\int_0^{\pi/2} \frac{r^2}{2}\Big]_{r=\cos\theta}^{r=2} d\theta = \delta\int_0^{\pi/2} \frac{2^2}{2} - \frac{\cos^2\theta}{2}\, d\theta$$

$$= 2\delta\frac{\pi}{2} - \frac{1}{2}\delta\int_0^{\pi/2}\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)}\, d\theta$$

$$= \delta\pi - \frac{1}{4}\delta\int_0^{\pi/2} 1+\cos 2\theta\, d\theta$$

$$= \delta\pi - \frac{1}{4}\delta\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2}$$

$$= \delta\pi - \delta\frac{\pi}{8} - \underbrace{\frac{1}{8}\sin 2\theta\Big]_0^{\pi/2}}_{0} = \delta\frac{7\pi}{8}$$

(12)

☒ Set up the integral in polar coords for radius of gyration about the x-axis:

<u>Soln</u>: Rad Gyration $= \sqrt{\dfrac{I_x}{M}}$

$$I_x = \iint_R y^2 \sigma \, dA$$

$$= \iint_{R_{r\theta}} (r\sin\theta)^2 \sigma \, r \, dr \, d\theta$$

$$= \sigma \int_0^{\pi/2} \int_{\cos\theta}^{2} r^3 \sin^2\theta \, dr \, d\theta$$

$$M = \frac{7\pi}{8}\sigma$$

✓

Important example —

- The "bell shaped curve" of probabity theory is called the **Gaussian Distribution**

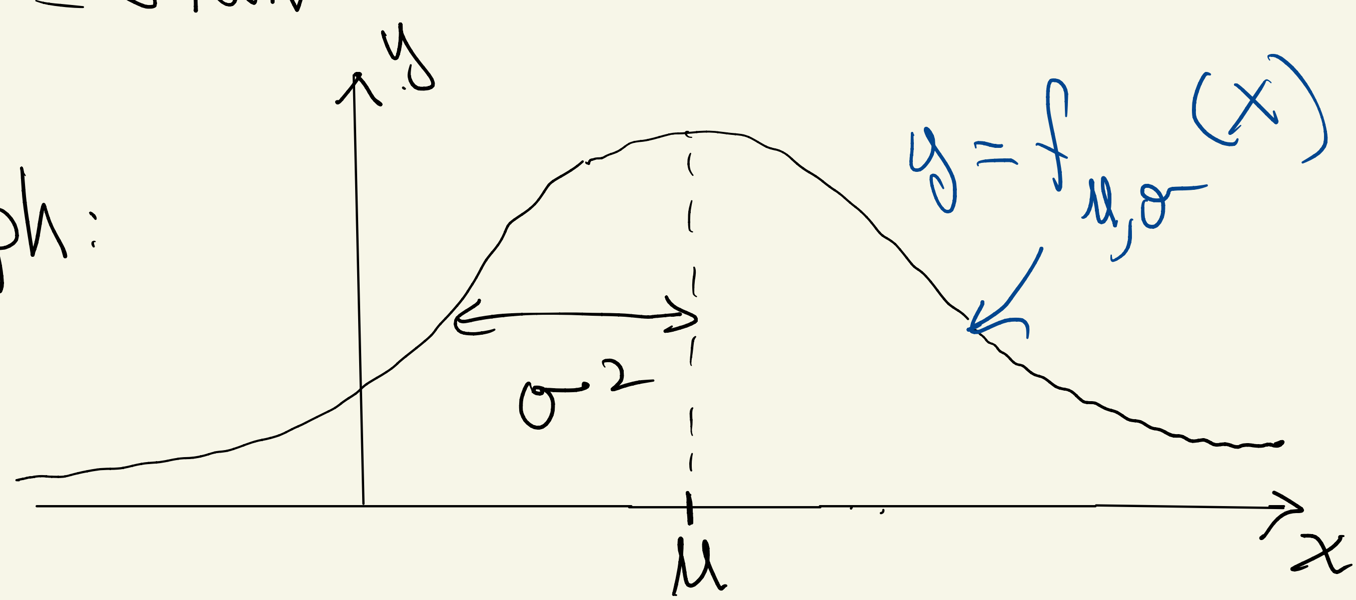
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = f_{\mu,\sigma}(x)$$

μ = mean

σ^2 = variance

σ = standard deviation

Graph:

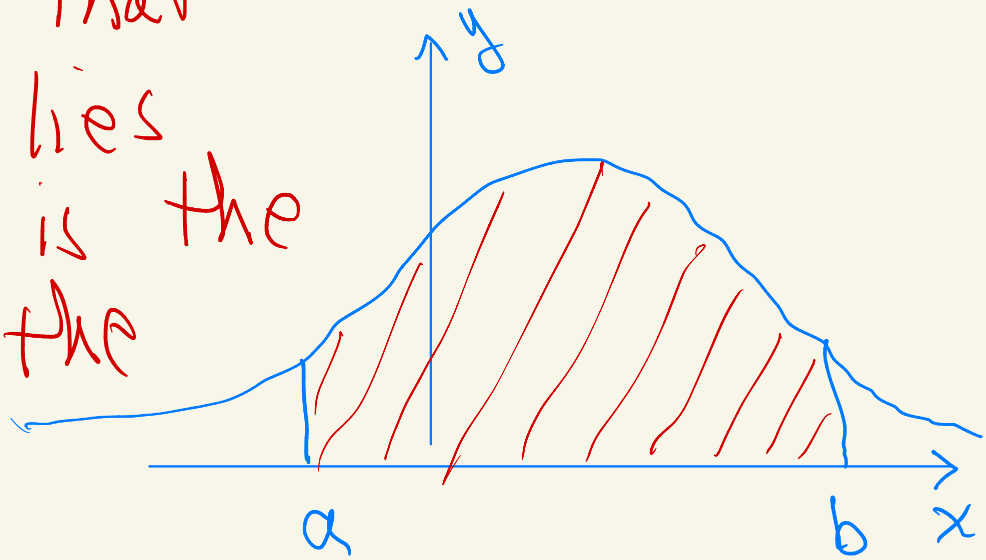


- **Theorem**: The average of N outcomes of a random variable (appropriately rescaled) <u>always</u> tends to $f_{\mu,\sigma}$ for some μ, σ

- Background: in the modern theory of probability (Kolmagorov)

$$\text{Prob}\{x \in [a,b]\} = \int_a^b f_{\mu,\sigma}(x)\,dx$$

"The probabity that the outcome lies betw a & b is the area under the graph"



- Probability is a number between zero and one - So for the theory to make sense, we must have

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = 1$$

Problem: Prove This?

Soln: simplify and evaluate in polar coordinates

- **Change Variables**

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2}$$

So...

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2} dx$$

Set: $u = \frac{x-\mu}{\sqrt{2}\,\sigma}$

$$du = \frac{dx}{\sqrt{2}\,\sigma}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

Conclude: it suffices to show

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

or

$$\int_{0}^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$$

Problem - there is no Math 21B substitution that works!

For example: $V = u^2 \Rightarrow dv = 2u\,du$

Doesn't work! ??

The new trick employs polar coordinates —

Set $I = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dy\,dx$

$$= \int_0^\infty e^{-x^2} \underbrace{\left[\int_0^\infty e^{-y^2}\,dy\right]}_{\text{constant !}} dx$$

$$= \left[\int_0^\infty e^{-y^2}\,dy\right] \int_0^\infty e^{-x^2}\,dx$$

same integrals

$$= \left(\int_0^\infty e^{-x^2}\,dx\right)^2 \Rightarrow \int_0^\infty e^{-x^2}\,dx = \sqrt{I}$$

Thus to evaluate:

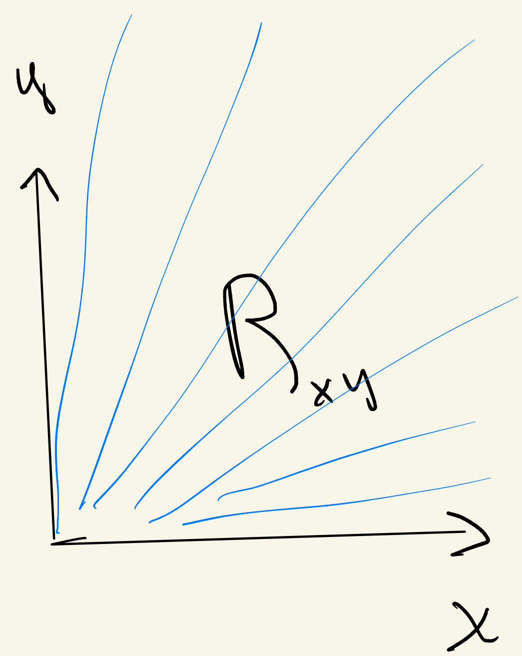
$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

polar coordinates: $r^2 = x^2 + y^2$

$$R_{xy}: [0,\infty] \times [0,\infty]$$

$$R_{r\theta}: \quad 0 \le r \le \infty, \quad 0 \le \theta \le \pi/2$$

thus



$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta$$

$$I = \int_0^{\pi/2}\underbrace{\int_0^{\infty} e^{-r^2}\, r\, dr}\, d\theta$$

$u = r^2 \quad du = 2r\,dr$

$$= \int_0^{\pi/2} \frac{1}{2}\int_0^{\infty} e^{-u}\, du$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[-e^{-u}\right]_{u=0}^{u=\infty} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{1}{2}\theta\Big]_0^{\pi/2}$$

$$= \pi/4 \qquad \int_0^{\infty} e^{-u^2} du = \sqrt{I} = \frac{\sqrt{\pi}}{2}$$

✓